Partial Differential Equations - Final exam

You have 3 hours to answer the following questions, and 20 minutes of scan time. Everyone receives 10 points for turning in their exam. This exam is open book and open note, but not open internet. Justify all work with appropriate theorems. Late answers will not be accepted.

(1) (25 points total)
  (a) (15 points) Use the method of images to find formally the Green's function for
$$\Delta u = f \quad \text{in} \quad \Omega$$
$$u = g \quad \text{on} \quad \partial\Omega$$
where Ω is the upper half plane $\{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty, y > 0\}$.
  (b) (5 points) Write down the solution u in terms of f and g.
  (c) (5 points) if instead Ω is a unit ball and f is positive everywhere, what is the maximum value of u and where is it obtained?

(2) (15 points) Consider the function $f(x) = x^3$.
  (a) (6 points) Compute the Fourier series of f on the interval $[-\pi, \pi]$.
  (b) (4 points) Draw and write down explicitly to what function the Fourier series converges.
  (c) (5 points) Does it converge pointwise? Does it converge uniformly? Justify the type of convergence by stating the appropriate theorem.

(3) (30 points total) Consider the following PDE
$$\partial_t u = \Delta u + \alpha u$$
  (a) (10 points) For which values of $\alpha \in \mathbb{R}$ do we have uniqueness of the solution on $\mathbb{R}_+ \times (0, 1)$ with $u(t, 0) = u(t, 1) = 0$?
  (b) (10 points) With the value $\alpha = -1$, solve the PDE using separation of variables with the initial condition
$$u(0, x) = 3\sin(\pi x) + \frac{1}{7}\sin(3\pi x)$$
and boundary conditions as above.
  (c) (10 points) Solve the PDE using the Fourier transform with $\alpha < 0$ on $\mathbb{R}_+ \times \mathbb{R}$ with $u(0, x) = e^{-|x|}$. You may leave your answer in terms of an inverse Fourier transform. What happens to the maximum of your solution as $t \to \infty$?

(4) (20 points total)
  (a) (10 points) Classify the following partial differential equation
$$2u_{xx} - 4u_{xy} - u_{yy} = 0$$
and put it in canonical form using the variables (ξ, η).
  (b) (10 points) Solve the partial differential equation with $u(\xi, 0) = e^{-\xi^2}$ and $\partial_\eta u(\xi, 0) = 0$ and write your solution in terms of the original variables (x, y).